Presentation to East Coast Energy Group

"Dynamic Theories of Oligopoly"

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Folk Theorem

Stage game

$A_i \equiv$ (pure) action set of player i for the stage game

$v_i : A_1 \times \cdots \times A_n \rightarrow \Re \equiv$ payoff function of player i for the stage game

Infinitely repeated game

A strategy is of the form: $\{f_i^t\}_{t=1}^{\infty}$ where $f_i^t : \Pi_{j=1}^{n} A_i^{t-1} \rightarrow A_i$

Payoff is the sum of discounted single-period utilities where δ is player $i's$ discount factor

$$\sum_{t=1}^{\infty} \delta^{t-1} v_i \left(a_1^t, \ldots, a_n^t\right)$$

Minimax

M_{-i} are strategies of the other $n-1$ players that minimize player $i's$ maximum payoff

$M_{-i} \in \arg\min_{a_{-i}} \max_{a_i} v_i(a_i, a_{-i})$ where $a_{-i} \in \Pi_{j \neq i} A_j$

$v_i^* = \max_{a_i} v_i(a_i, M_{-i})$

Set of individually rational payoffs

$(v_1, ..., v_n)$ is individually rational iff $v_i \geq v_i^* \forall i$

$U \equiv \{(v_1, ..., v_n) | \exists (a_1, ..., a_n) \in A_1 \times \cdots \times A_n \text{ with } v_i(a_i, a_{-i}) = v_i \forall i\}$

$V \equiv$ convex hull of U (smallest convex set containing U)

$V^* \equiv \{(v_1, ..., v_n) \in V | v_i > v_i^* \forall i\}$

Folk Theorem: For all $(v_1, ..., v_n) \in V^*$, if δ is sufficiently close to one then there exists a Nash equilibrium such that the average payoff is $v_i \forall i$.

Review of Static Quantity Game

Structure

$n \geq 2$ firms have homogeneous products.

Firms make simultaneous quantity decisions

Price is set in the market so as to equate supply and demand.

Assumptions on the Inverse Market Demand Function

A1 $P(\cdot) : \Re_+ \rightarrow \Re_+$ is continuous and bounded $\forall Q \geq 0$.

A2 $\exists$ finite $\overline{Q} > 0$ such that $P(Q) = 0$ iff $Q \geq \overline{Q}$.

A3 $P(\cdot)$ is twice differentiable and $P'(Q) < 0 \; \forall Q \in \left(0, \overline{Q}\right)$.

Assumptions on the Firm Cost Function

A4 $C_i(\cdot) : \Re_+ \rightarrow \Re_+$ is continuous $\forall q \geq 0$.

A5 $C_i(\cdot)$ is twice differentiable and $C_i'(q) \geq 0 \forall q > 0$.

Implied Structure on the Firm Profit Function by Assumptions A1-A5

$\pi_i(q_i, Q_{-i}) \equiv P(q_i + Q_{-i}) q_i - C_i(q_i)$

$\pi_i(\cdot)$ is continuous and bounded from above $\forall q_i, Q_{-i} \geq 0$

$\pi_i(\cdot)$ is twice differentiable in q_i and $Q_{-i} \forall (q_i, Q_{-i}) \in \{(q_i, Q_{-i}) : q_i + Q_{-i} \in (0, \overline{Q})\}$.

Theorem 1 (Existence of Best Response Function): By A1-A5, $\exists \psi_i(\cdot) : \Re_+ \to \Re_+$ such that:

$$\pi_i(\psi_i(Q_{-i}), Q_{-i}) \geq \pi_i(q_i, Q_{-i}) \forall q_i \in [0, \overline{Q}], \forall Q_{-i} \geq 0.$$

A6 $\pi_i(\cdot)$ is strictly quasi-concave in q_i.

Theorem 2 (Continuity of the Best Response Function): By A1-A6, $\psi_i(\cdot)$ is a continuous function $\forall Q_{-i} \geq 0$.

Theorem 3 (Existence of Nash Equilibrium): By A1-A6, $\exists (\widehat{q}_1, ..., \widehat{q}_n) \in [0, \overline{Q}]^n$ such that

$$\widehat{q}_i = \psi_i\left(\sum_{j \neq i} \widehat{q}_j\right) \ \forall i \in \{1, ..., n\}.$$

A7 $P''(Q) \leq 0 \; \forall Q \in \left(0, \overline{Q}\right)$.

A8 $C_i''(q) \geq 0 \; \forall q > 0$.

By A1-A8, $\pi_i(\cdot)$ is strictly concave in $q_i \; \forall q_i \in \left\{\left(q_i^o, Q_{-i}^o\right) : q_i^o + Q_{-i}^o \in \left(0, \overline{Q}\right)\right\}$.

Theorem 4 (Best Response Function is Decreasing): By A1-A5 and A7-A8,

i) $\psi_i(\cdot)$ is differentiable in $Q_{-i} \; \forall Q_{-i} \in \left\{Q_{-i}^o : Q_{-i}^o > 0 \text{ and } \psi_i\left(Q_{-i}^o\right) > 0\right\}$;

ii) if $\psi(Q_{-i}) > 0$ then $\psi'(Q_{-i}) < 0$.

Theorem 5 (Uniqueness of Nash Equilibrium): By A1-A5 and A7-A8, $\exists$ a unique solution $(\widehat{q}_1, ..., \widehat{q}_n) \in \left[0, \overline{Q}\right]^n$ to the n-equation system:

$$\widehat{q}_i = \psi_i\left(\sum_{j \neq i} \widehat{q}_j\right) \forall i \in \{1, ..., n\}.$$

A9 $C_i(q) = C_j(q) \; \forall q \geq 0, \forall i, j \in \{1, ..., n\}$.

Theorem 6 (Existence of Symmetric Nash Equilibrium): By A1-A5 and A7-A9, $\exists \widehat{q} \in \left[0, \frac{\overline{Q}}{n}\right]$ such that $\widehat{q} = \psi((n-1)\widehat{q})$.

A10 $P(0) > C_i'(0)$.

Theorem 7 (Existence of an Interior Symmetric Nash Equilibrium): By A1-A5 and A7-A10, $\exists \widehat{q} \in \left(0, \frac{\overline{Q}}{n}\right)$ such that $\widehat{q} = \psi\left((n-1)\,\widehat{q}\right)$.

Infinitely Repeated Symmetric Quantity Game

- Friedman, *Review of Economic Studies*, 1971

Strategic Form

Set of players/firms is $\{1, \ldots, n\}$

Strategy set of firm i is the set of functions of the following form:

$$\{f_i^t\}_{t=1}^{\infty} \text{ where } f_i^t : \Omega_i^t \to A_i$$

Ω_i^t is the set of information sets of firm i in period t

A_i is the set of quantities available to firm i

Informational assumptions

If all past quantities are common knowledge then

$$\Omega_i^t \equiv \Pi_{j=1}^n A_j^{t-1} \forall i$$

If each firm knows its own past quantities and past prices then

$$\Omega_i^t \equiv A_i^{t-1} \times \Re_+^{t-1} \forall i$$

Payoff is the sum of discounted single-period profits:

$$\sum_{t=1}^{\infty} \delta^{t-1} \pi_i \left(q_i^t, Q_{-i}^t\right) \text{ where } Q_{-i}^t \equiv \sum_{j \neq i} q_j^t$$

$\pi_i \left(q_i^t, Q_{-i}^t\right) \equiv P\left(q_i^t + Q_{-i}^t\right) q_i^t - C\left(q_i^t\right)$ (Homogeneous products)
$\delta_i \in (0, 1)$

Some stage game outcomes

$\widehat{q}$ is a static Nash equilibrium quantity:

$$\widehat{q} \in \arg\max \pi\left(q, (n-1)\widehat{q}\right)$$

q^m is a joint profit-maximizing quantity:

$$q^m \in \arg\max \pi(q, (n-1)q)$$

Grim trigger strategy:

$$f_i^1 = q^o \tag{1}$$

$$f_i^t = \begin{cases} q^o \text{ if } q_j^\tau = q^o \forall \tau \leq t-1, \forall j \\ \\ \widehat{q} \text{ otherwise; } t \geq 2, i = 1, \ldots, n \end{cases}$$

$q^o \in (q^m, \widehat{q}]$

Definitions

$$\pi(q) \equiv P(nq)q - C(q)$$

$$\widehat{\pi} \equiv \pi(\widehat{q})$$

$$\pi^*(q) \equiv P(\psi((n-1)q) + (n-1)q)\psi((n-1)q) - C(\psi((n-1)q)) \text{ where}$$

$$\psi\left(Q_{-i}\right) \in \arg\max \pi_i \left(q_i^t, Q_{-i}^t\right)$$

A *subgame perfect equilibrium* is a strategy profile which forms a Nash equilibrium in every subgame.

Necessary and sufficient conditions for this strategy profile to be a subgame perfect equilibrium.

1) Consider period 1 or a period t history such that $q_j^\tau = q^o \forall \tau \leq t-1, \forall j$. SPE requires:

$$\frac{\pi(q^o)}{1-\delta_i} \geq \pi(q,(n-1)q^o) + \frac{\delta_i \widehat{\pi}}{1-\delta_i} \ \forall q \Leftrightarrow \tag{2}$$

$$\frac{\pi(q^o)}{1-\delta_i} \geq \pi^*(q^o) + \frac{\delta_i \widehat{\pi}}{1-\delta_i} \Leftrightarrow \delta_i \geq \frac{\pi^*(q^o)-\pi(q^o)}{\pi^*(q^o)-\widehat{\pi}}$$

2) Consider a period t history such that $q_j^\tau \neq q^o$ for some $\tau \leq t-1$ and for some j

$$\frac{\widehat{\pi}}{1-\delta_i} \geq \pi(q,(n-1)\widehat{q}) + \frac{\delta_i \widehat{\pi}}{1-\delta_i} \ \forall q \tag{3}$$

This strategy profile is a SPE iff:

$$\delta_i \geq \frac{\pi^*(q^o)-\pi(q^o)}{\pi^*(q^o)-\widehat{\pi}} \forall i \Leftrightarrow \min\{\delta_1,...,\delta_n\} \geq \frac{\pi^*(q^o)-\pi(q^o)}{\pi^*(q^o)-\widehat{\pi}} \tag{4}$$

Example with Linear Demand and Cost

Linear inverse market demand curve:

$$P(Q) = a - bQ$$

where $a, b > 0$.

Linear firm cost function,

$$C_i(q) = cq$$

where $0 \leq c < a$.

Firm profit function

$$\pi(q_i, Q_{-i}) = [a - b(q_i + Q_{-i}) - c]\, q_i$$

where $Q_{-i} = \sum_{j \neq i} q_j$.

Best reply function

$$\psi(Q_{-i}) = \frac{a-c}{2b} - \frac{Q_{-i}}{2}.$$

Static Nash Equilibrium

$$\widehat{q} = \psi((n-1)\widehat{q}) \Leftrightarrow \widehat{q} = \frac{a-c}{2b} + \frac{(n-1)\widehat{q}}{2} \Leftrightarrow \widehat{q} = \frac{a-c}{b(n+1)}$$

$$\widehat{\pi} \equiv \frac{(a-c)^2}{b\,(n+1)^2}$$

Equilibrium condition:

$$\frac{\pi(q^o)}{1-\delta} \geq \pi^*(q^o) + \frac{\delta\widehat{\pi}}{1-\delta} \Leftrightarrow \pi(q^o) \geq (1-\delta)\,\pi^*(q^o) + \delta\widehat{\pi} \tag{5}$$

where

$$\begin{aligned}
\pi(q) &\equiv (a - bnq^o - c)\,q \\
\pi^*(q) &\equiv [a - b\,(\psi\,((n-1)\,q) + (n-1)\,q) - c]\,\psi\,((n-1)\,q) = \frac{(a-c-b\,(n-1)\,q)^2}{4b}.
\end{aligned}$$

Using the closed-form solutions, the condition is

$$\frac{(a - bnq^o - c)\,q^o}{1-\delta} \geq \frac{(a-c-b\,(n-1)\,q^o)^2}{4b} + \left(\frac{\delta}{1-\delta}\right)\frac{(a-c)^2}{b\,(n+1)^2}.$$

Simple Strategy Profiles

- Abreu, *Econometrica*, 1988

Definitions

A_i is the stage game action set for player i.

$Q^j \in (A_1 \times \cdots \times A_n)^\infty \equiv \Omega$ is an outcome path for the infinitely repeated game

$\Omega^o \equiv$ set of SPE outcome paths.

Definition: $\sigma(Q^0, Q^1, ..., Q^n)$ is a *simple strategy profile* if

1. players play according to Q^o until some player deviates from that outcome path
2. for any $j \in \{1, ..., n\}$, players play according to Q^j (starting with the first element) when player j deviates from the current path
3. if two or more players simultaneously deviate then players play according to the current outcome path

Theorem: $Q^o \in \Omega^o$ iff $\exists Q^i \in \Omega \forall i$ such that $\sigma(Q^0, Q^1, ..., Q^n)$ is a SPE.

Most Severe Punishment Strategy Equilibria in the Infinitely Repeated Quantity Game

- Abreu, *Journal of Economic Theory*, 1986

Assumptions

Stage game is two-firm quantity game with homogeneous goods and constant marginal cost, c

$P(\cdot) : \Re_+ \rightarrow \Re_+$ is strictly monotonic and continuous

$P(0) > c > 0$

$\pi(q) \equiv P(nq)q - cq$ is strictly quasi-concave in q with a maximum of q^m

The stage game has a symmetric pure-strategy Nash equilibrium

Statement of problem

Definitions

Γ is the set of subgame perfect equilibria such that, for every history, the outcome path is symmetric

$v_i(\gamma)$ is the payoff to player i from strategy profile γ

Problem A: Find $\gamma^* \in \Gamma$ such that $v_i(\gamma^*) \geq v_i(\gamma)\, \forall \gamma \in \Gamma$.

Definition: $\sigma(Q^0, Q^1, ..., Q^n)$ is a *simple strategy profile* if

1. players play according to Q^o until some player deviates from that outcome path
2. for any $j \in \{1, ..., n\}$, players play according to Q^j (starting with the first element) when player j deviates from the current path
3. if two or more players simultaneously deviate then players play according to the current outcome path

Stick-and-carrot simple strategy profile

$Q^o = \{(q^o, q^o), ...\}$ (initial collusive phase)
$Q^1 = Q^2 = \{(\bar{q}, \bar{q}), (q^o, q^o), ...\}$ (punishment phase)

SPE conditions

Collusive stage (Q^o; τ^{th} element of Q^1 or $Q^2, \tau \geq 2$)

$$\frac{\pi(q^o)}{1-\delta} \geq \pi^*(q^o) + \delta\pi(\bar{q}) + \delta^2 \left[\frac{\pi(q^o)}{1-\delta}\right] \Leftrightarrow \delta \geq \frac{\pi^*(q^o) - \pi(q^o)}{\pi(q^o) - \pi(\bar{q})} \qquad (6)$$

Punishment stage (first element of Q^1 or Q^2)

$$\pi(\bar{q}) + \delta \left[\frac{\pi(q^o)}{1-\delta}\right] \geq \pi^*(\bar{q}) + \delta\pi(\bar{q}) + \delta^2 \left[\frac{\pi(q^o)}{1-\delta}\right] \Leftrightarrow \delta \geq \frac{\pi^*(\bar{q}) - \pi(\bar{q})}{\pi(q^o) - \pi(\bar{q})} \qquad (7)$$

Optimal stick-and-carrot strategy (a solution to Problem A)

Stick-and-carrot strategy in which $(\bar{q}, q^o)$ satisfies

$$\pi(\overline{q}) + \delta\pi(q^o) = \pi^*(\overline{q}) + \delta\pi(\bar{q}) \tag{7'}$$

$$\pi(q^o) + \delta\pi(q^o) = \pi^*(q^o) + \delta\pi(\bar{q}) \text{ if } q^o \neq q^m \tag{6'}$$

$$\pi(q^o) + \delta\pi(q^o) \geq \pi^*(q^o) + \delta\pi(\bar{q}) \text{ if } q^o = q^m \tag{6''}$$

If $\pi(\cdot)$ and $\pi^*(\cdot)$ are continuously differentiable then $\bar{q} > \widehat{q} > q^o$.

Collusion with Imperfect Monitoring

- Porter, *Journal of Economic Theory*, 1983

Model

Demand: $P^t = \theta^t P\left(Q^t\right) = \theta^t (a - bQ^t)$

θ^t is an *iid r.v.* with cdf $F(\cdot)$

$F(0) = 0$, $F(\theta^o) = 1$, $\theta^o < \infty$

$F(\cdot)$ is continuously differentiable and convex

Cost: $C(q) = c_o + c_1 q$

Information and strategy sets

A firm knows all past prices and all of its past quantities

Only past prices are common knowledge

A strategy is an infinite sequence of functions in which the period t function maps from $\Re_+^{2(t-1)}$ into $\Re_+$.

Trigger strategies with imperfect monitoring

If in the *cooperative phase* in period $t-1$ and

$P^{t-1} \geq \widetilde{P}$ then $q_i^t = q^o$ and remain in the cooperative phase

$P^{t-1} < \widetilde{P}$ then $q_i^t = \hat{q}$ and go to the punishment phase

If in the τ^{th} period of the *punishment phase* in period $t-1$ and

$\tau < T$ then $q_i^t = \hat{q}$ and remain in the punishment phase

$\tau \geq T$ then $q_i^t = q^o$ and go to the cooperative phase

Subgame perfect equilibrium conditions

Notation

$\pi(q_i, Q_{-i}) \equiv \int [\theta P(q_i + Q_{-i}) - c_o - c_1 q_i] F'(\theta)\, d\theta$

$\pi(q) \equiv \int [\theta P(nq) - c_o - c_1 q] F'(\theta)\, d\theta$

$\widehat{q} \in \arg\max \pi(q, (n-1)\widehat{q})$ (static Nash equilibrium quantity)

$q^o \equiv$ generic collusive quantity

$q^* \equiv$ equilibrium collusive quantity

$V(q^o)$ is a firm's payoff when firms are in the cooperative phase and the collusive quantity is q^o

$$V(q^o) = \pi(q^o) + \left[1 - F\left(\frac{\widetilde{P}}{P(nq^o)}\right)\right]\delta V(q^o) + F\left(\frac{\widetilde{P}}{P(nq^o)}\right)\left[\sum_{\tau=1}^{T-1}\delta^{\tau}\pi(\widehat{q}) + \delta^T V(q^o)\right]$$

$$V(q^o) = \frac{\pi(\hat{q})}{1-\delta} + \frac{\pi(q^o) - \pi(\widehat{q})}{1 - \delta + (\delta - \delta^T)F\left(\frac{\widetilde{P}}{P(nq^o)}\right)} \tag{8}$$

First-order condition on q^*

$$\begin{aligned} q^* \ \in \ & \arg\max \pi\left(q, (n-1)\, q^*\right) + \left[1 - F\left(\frac{\widetilde{P}}{P\left(q + (n - 1q^*)\right)}\right)\right]\delta V\left(q^*\right) \\ & + F\left(\frac{\widetilde{P}}{P\left(q + (n-1q^*)\right)}\right)\left[\left(\frac{\delta - \delta^T}{1-\delta}\right)\pi\left(\widehat{q}\right) + \delta^T V\left(q^*\right)\right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \cdot}{\partial q_i} \ &= \ 0 = \frac{\partial \pi\left(q^*, (n-1)\, q^*\right)}{\partial q_i} + \\ & F'\left(\frac{\widetilde{P}}{P\left(q^* + (n-1q^*)\right)}\right)\left[\frac{\widetilde{P}P'\left(nq^*\right)}{P\left(nq^*\right)^2}\right]\left\{\delta V\left(q^*\right) - \left(\frac{\delta - \delta^T}{1-\delta}\right)\pi\left(\widehat{q}\right) - \delta^T V\left(q^*\right)\right\} \end{aligned} \tag{9}$$

Using (8), substitute for $V(q^*)$:

$$\frac{\partial \cdot}{\partial q_i} = 0 = \frac{\partial \pi \left(q^*, (n-1) q^*\right)}{\partial q_i} + F'\left(\frac{\widetilde{P}}{P\left(q^* + (n-1q^*)\right)}\right)\left[\frac{\widetilde{P}P'\left(nq^*\right)}{P\left(nq^*\right)^2}\right] \times$$

$$\left\{\left(\delta - \delta^T\right)\left[\frac{\pi(\hat{q})}{1-\delta} + \frac{\pi(q^*) - \pi(\hat{q})}{1 - \delta + \left(\delta - \delta^T\right) F\left(\frac{\widetilde{P}}{P(nq^*)}\right)}\right] - \left(\frac{\delta - \delta^T}{1-\delta}\right)\pi\left(\widehat{q}\right)\right\}$$

$$0 = \frac{\partial \pi \left(q^*, (n-1) q^*\right)}{\partial q_i} + \left(\delta - \delta^T\right) F'\left(\frac{\widetilde{P}}{P\left(q^* + (n-1q^*)\right)}\right) \times$$

$$\left[\frac{\widetilde{P}P'\left(nq^*\right)}{P\left(nq^*\right)^2}\right]\left[\frac{\pi\left(q^*\right) - \pi\left(\widehat{q}\right)}{1 - \delta + \left(\delta - \delta^T\right) F\left(\frac{\widetilde{P}}{P(q^* + (n-1q^*))}\right)}\right]$$

A most severe punishment strategy (Abreu, Pearce, and Stachetti, *Journal of Economic Theory*, 1986)

If in the *cooperative phase* in period $t-1$ and

$P^{t-1} \geq \bar{P}$ then $q_i^t = q^o$ and remain in the cooperative phase

$P^{t-1} < \bar{P}$ then $q_i^t = \overline{q}$ and go to the punishment phase

If in the *punishment phase* in period $t-1$ and

$P^{t-1} \leq \underline{P}$ then $q_i^t = q^o$ and go to the cooperative phase

$P^{t-1} > \underline{P}$ then $q_i^t = \bar{q}$ and remain in the punishment phase

Example with Demand and Cost Shocks

Cost and demand conditions

Standard infinitely repeated quantity game except that demand and cost functions are subject to observable shocks

$$P_t(Q) = a_0 + a_1 x_t - a_2 Q$$

$$C_t(q) = (c_0 + c_1 w_t)\, q$$

where $a_0, a_1, a_2, c_0, c_1 > 0$.

$(x_t, w_t) \in X \times W$ are *iid* demand and cost shifters that are observable in period t prior to firms choosing quantity.

$\pi(q; x_t, w_t) \equiv P(nq; x_t)\, q - C(q; w_t)$

$\pi^*(q; x_t, w_t) = \max_{q_i} P(q_i + (n-1)\, q; x_t)\, q_i - C(q_i; w_t)$

$\widehat{\pi}(x_t, w_t)$ is the static Nash equilibrium profit

Punishment strategy equilibrium (grim trigger)

Punishment is infinite reversion to static Nash equilibrium.

Collusive quantity is chosen to maximize profit subject to the incentive compatibility constraints (no firm has an incentive to deviate)

Optimal collusive quantity, $q^*(x_t, w_t) : X \times W \rightarrow \Re_+$

Statement of problem:

$$\max_q \pi(q; x_t, w_t) \text{ subject to} \tag{10}$$

$$\pi(q; x_t, w_t) + \sum_{\tau=1}^{\infty} \delta^\tau E_t \left[\pi\left(q^*(x_{t+\tau}, w_{t+\tau}); x_{t+\tau}, w_{t+\tau}\right)\right]$$
$$\geq \pi^*(q; x_t, w_t) + \sum_{\tau=1}^{\infty} \delta^\tau E_t \left[\widehat{\pi}(x_{t+\tau}, w_{t+\tau})\right], \ \forall (x_t, w_t)$$

Suppose δ is sufficiently low so that the constraint is binding $\forall (x_t, w_t)$.

Then the optimal collusive quantity is the lowest value that satisfies the constraints which is

$$q^*(x_t, w_t) = \widehat{q}(x_t, w_t) - \frac{2\sqrt{a_2 L}}{a_2(N+1)} \tag{11}$$

where $\widehat{q}(x_t, w_t)$ is the static Nash equilibrium quantity

$$\widehat{q}(x_t, w_t) = \frac{a_0 + a_1 x_t - c_0 - c_1 w_t}{a_2(N+1)}$$

where

$$L \equiv \sum_{\tau=1}^{\infty} \delta^{\tau} E_t \left[\pi \left(q^* \left(x_{t+\tau}, w_{t+\tau} \right); x_{t+\tau}, w_{t+\tau} \right) - \widehat{\pi} \left(x_{t+\tau}, w_{t+\tau} \right) \right]$$

is the loss due to deviation.

Extensions

- Demand movements
  - Anticipated demand movements (e.g., seasonal cycle)
    - Haltiwanger and Harrington, *RAND Journal of Economics*, 1991
  - Random demand movements
    - Rotemberg and Saloner, *American Economic Review*, 1986
- Multi-market collusion
  - Bernheim and Whinston, *RAND Journal of Economics*, 1990
- Firm asymmetries and deriving a unique solution
  - Joint profit maximization
  - Nash bargaining solution (Harrington, Hobbs, Pang, Liu, and Roch, 2003)

$$\max_{(q_1,\ldots,q_n)\in\Omega} \prod_{i=1}^{n} \left[\pi_i\left(q_i, Q_{-i}\right) - \widehat{\pi}_i\right]$$

where

$$\Omega \equiv \left\{\left(q_1^o, \ldots, q_n^o\right) \in \Re_+^n : \pi_i\left(q_i^o, Q_{-i}^o\right) \geq \left(1-\delta_i\right)\pi_i^*\left(Q_{-i}^o\right) + \delta_i\widehat{\pi}_i, \forall i\right\}.$$