Notes on joint equilibrium of day-ahead and real-time markets

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Abstract

We consider a day-ahead electricity market where a central operator forecasts the day-ahead demands and where there may be some price elasticity. The day-ahead market clears based on the day-ahead bids. There is also a real-time deviations market, which clears the difference between the day-ahead demand and the real-time demand, which is modeled as a random variable. We observe that in the absence of arbitrage, the expected value of the real-time price is lower than the day-ahead price.

1 Model

Let t represent the time in the day. We assume that there is a day-ahead forecast of demand (and of price responsiveness of demand) by a central operator of the form:

$$D(p^D, t) = N(t) - \gamma^D p^D,$$

where p^D is the day-ahead price and γ^D is the (possibly zero-valued) demand slope of demand scheduled day-ahead.

We assume that generation firms $i = 1, \ldots, n$ have quadratic total cost functions of the form:

$$C_i(q_i) = \frac{1}{2} c_i (q_i)^2 + a_i q_i,$$

and marginal cost functions of the form:

$$C_i'(q_i) = c_i q_i + a_i.$$

We assume that the firms bid day-ahead supply functions $S_i, i = 1, \ldots, n$, that apply throughout the day and that, for each time, the day-ahead market is cleared to obtain quantities $q_i(t), i = 1, \ldots, n$ and prices $p^D(t)$.

In real-time, however, the demand turns out to be different to the cleared day-ahead demand. At each time t, the deviation of the demand from the cleared day-ahead demand is modeled by the random variable $G(t)$, so that G is a stochastic process describing the forecast error. The analysis can be easily extended to the case where there is some demand responsiveness in real-time; however, with this extension the interpretation of the stochastic process representing the deviation of real-time from day-ahead demand becomes less straightforward. (In practice, real-time markets are not cleared "continuously" in time as we model, but we will assume that there are enough pricing periods so that a continuous time approximation is reasonable.)

For convenience in the calculations that follow, we assume that the forecast demand and price responsiveness is unbiassed in the sense that, for each t, $G(t)$ has conditional expectation zero, given the realization of cleared demand up to time t in the day-ahead market. We write $\mathcal{E}_t^D$ and Var_t^D for the conditional expectation and variance operaters, given the realization of cleared demand up to time t. We also assume that the conditional variance of $\text{Var}_t^D G(t)$ is a constant σ^2, independent of t. We do not model arbitrage between the day-ahead and real-time markets.

2 Analysis of real-time market

We now analyze the real-time market, given the cleared demand, the quantities $q_i(t), i = 1, \dots, n$, and the price $p^D(t)$ for each time t in the day-ahead market. We consider the change in cost to firm i of being called on to deliver a deviation, $\Delta q_i(t)$, from the quantity $q_i(t)$ cleared in the day-ahead market. This change in production cost is given by:

$$\frac{1}{2}c_i(\Delta q_i(t))^2 + (c_i q_i(t) + a_i)\Delta q_i(t),$$

and the marginal cost for the deviation $\Delta q_i(t)$ is given by:

$$c_i \Delta q_i(t) + (c_i q_i(t) + a_i).$$

At each t, and given $q_i(t)$, we can consider the equilibrium in the real-time market, assuming that all players make bids into the real-time market. This equilibrium is given by the solution of the supply function equilibrium corresponding to these marginal cost functions and the real-time demand.

Using the stability argument from Baldick and Hogan, we observe that since the marginal costs are linear then in the case of symmetric cost functions all SFEs besides the affine SFE are unstable. In our case, the cost functions are not necessarily symmetric; however, we will now assume that the real-time market equilibrium turns out to be the affine SFE.

The slope of the affine SFE depends on the values of c_i and not on the intercepts of the cost functions. We can calculate the slopes of the affine SFE for the real-time market and denote them by β_i^R, noting that they are independent of time. We also note that the intercept of the affine SFE is given by the "intercept" of the marginal cost function of the deviation cost:

$$c_i q_i(t) + a_i.$$

The deviation supply function $\Delta S_i(\bullet, t)$ offered into the real-time market at time t is therefore given by:

$$\forall p^R, \Delta S_i(p^R, t) = \beta_i^R(p^R - (c_i q_i(t) + a_i)).$$

We can now calculate the change in profit per unit time at each time t due to participating in the real-time market. Given a clearing price $p^R(t)$ at time t, the change in profit per unit time is:

$$\begin{aligned}
\Delta\Pi_i(t) &= \Delta S_i(p^R(t),t)p^R(t) - \frac{1}{2}c_i(\Delta S_i(p^R(t),t))^2 - (c_i q_i(t) + a_i)\Delta S_i(p^R(t),t), \\
&= \Delta S_i(p^R(t),t)[p^R(t) - (c_i q_i(t) + a_i) - \frac{1}{2}c_i \Delta S_i(p^R(t),t)], \\
&= (\Delta S_i(p^R(t),t))^2 \left[\frac{1}{\beta_i^R} - \frac{c_i}{2}\right], \text{ noting that } p^R - (c_i q_i(t) + a_i) = \frac{\Delta S_i(p^R,t)}{\beta_i^R}. \\
&> 0,
\end{aligned}$$

since $\frac{1}{\beta_i^R} \geq c_i$. Therfore, bidding in to the real-time market is always profitable, so our assumption that all players will participate in the real-time market is correct.

The price $p^R(t)$ in the real-time market is a random variable since the demand is random. Consequently, $\Delta S_i(p^R(t),t)$ is also random. In particular, the real-time market clearing condition $G(t) = \sum_{j=1}^n \Delta S_j(p^R(t),t)$ implies that:

$$\begin{aligned}
p^R(t) &= \frac{G(t) + \sum_{j=1}^n \beta_j^R (c_j q_j(t) + a_j)}{\sum_{j=1}^n \beta_j^R}, \\
&= \frac{G(t)}{\sum_{j=1}^n \beta_j^R} + \tau^\dagger q(t) + \mu, \\
\Delta S_i(p^R(t),t) &= \beta_i^R \left(\frac{G(t) + \sum_{j=1}^n \beta_j^R (c_j q_j(t) + a_j)}{\sum_{j=1}^n \beta_j^R} - (c_i q_i(t) + a_i) \right).
\end{aligned}$$

where:

- $q(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix} \in \mathbb{R}^n$ is the vector of day-ahead quantities,

- $\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = \begin{bmatrix} \frac{\beta_1^R c_1}{\sum_{j=1}^n \beta_j^R} \\ \vdots \\ \frac{\beta_n^R c_n}{\sum_{j=1}^n \beta_j^R} \end{bmatrix} \in \mathbb{R}^n$,

- $\mu = \frac{\beta_i^R a_i}{\sum_{j=1}^n \beta_j^R}$ is a weighted average of the a_i, and

- superscript $\dagger$ means transpose.

Substituting into the expression for $\Delta\Pi_i(t)$, we obtain:

$$\Delta\Pi_i(t) = \beta_i^R \left[1 - \frac{\beta_i^R c_i}{2}\right] \left(\frac{G(t)}{\sum_{j=1}^n \beta_j^R} + \tau^\dagger q(t) + \mu - c_i q_i(t) - a_i \right)^2.$$

3 Expected price in real time market

Given the assumptions about the distribution of $G(t)$, we have:

$$\mathcal{E}_t^D[p^R(t)] = \tau^\dagger q(t) + \mu,$$

which is a weighted average of the marginal costs of the firms at the day-ahead quantities. Assuming that the firms never offer energy at below their marginal cost, this implies that the real-time price is lower than the day-ahead price in expectation. Moreover, the day-ahead and real-time prices would be equal in expectation if the day-ahead market were competitive. (In the California power exchange, the opposite was observed; however, there was a binding price cap in the day-ahead market that was less stringently applied in real-time. In New York, in some zones the real-time prices appear to be above the the day-ahead prices on average while in other zones the opposite is true. Transmission constraints and the New York ISO unit commitment procedure apparently play a role in these effects.) Finally, we note that in the presence of arbitrage, the assumption about the forecast error having mean zero may no longer be true.

4 Representing profits from the real-time market into the day-ahead market

Let us suppose that the firms are risk neutral in considering the effects of the real-time market on their day-ahead decisions. Recall that $\mathcal{E}_t^D[G(t)] = 0$ and that $\mathrm{Var}_t^D[G(t)] = \sigma^2$. That is, we can use the conditional expectation of $\Delta\Pi_i(t)$ to assess the effect of the real-time market on the day-ahead decisions. We have that:

$$\begin{aligned}
\mathcal{E}^D(t)[\Delta\Pi_i(t)] &= \beta_i^R \left[1 - \frac{\beta_i^R c_i}{2}\right] \left(\frac{\sigma^2}{\sum_{j=1}^n \beta_j^R} + (\tau^\dagger q(t) + \mu - c_i q_i(t) - a_i)^2 \right), \\
&= \Delta\pi_i(q_i(t), q_j(t), j \neq i),
\end{aligned}$$

where $\Delta\pi_i : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and we note that the function $\Delta\pi_i$ is quadratic in the day-ahead quantities and has coefficients that are independent of t.

As noted above, the term $\tau^\dagger q(t) + \mu$ is a weighted average of the marginal costs of the firms at the production specified by their day-ahead quantities. If the firms are symmetric, or more generally if, at each time, the marginal costs at the day-ahead quantities are equal then $c_i q_i(t) + a_i = \tau^\dagger q(t) + \mu$ and $E_t^D[\Delta\Pi_i(t)]$ is constant independent of the day-ahead decisions. In this case, decisions in the real-time and day-ahead markets are decoupled and the equilibrium in the day-ahead market can be obtained independent of the equilibrium in the real-time market. We will consider the more general case where $c_i q_i(t) + a_i \neq \tau^\dagger q(t) + \mu$ in the next section.

5 Calculating the joint equilibrium in the real-time and day-ahead markets

If we ignored the real-time market, we could consider the standard SFE formulation for the day-ahead market. Suppose that each player $j \neq i$ has committed to a supply function S_j^D in the day-ahead market and that firm i has committed to supplying the difference between the demand and the supply of the other firms. We could evaluate the profit for firm i by:

$$\begin{aligned}
\pi_i(t) &= \text{revenues - costs}, \\
&= p^D(t)q_i(t) - \frac{1}{2}c_i(q_i(t))^2 - a_i q_i(t), \text{ where } q_i(t) = N(t) - \gamma^D p^D(t) - \textstyle\sum_{j\neq i} S_j^D(p^D(t)), \\
&= q_i(t)(p^D(t) - \frac{1}{2}c_i q_i(t) - a_i), \\
&+ \left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) \\
&\qquad \times \left(p^D(t) - \frac{1}{2}c_i\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) - a_i\right).
\end{aligned}$$

The effect of the real-time market, however, is to introduce an additional term, $\Delta\pi_i$, so that the profit becomes:

$$\begin{aligned}
&\pi_i(t) + \Delta\pi_i\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t)), S_j^D(p^D(t)), j \neq i\right) \\
&= \left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) \\
&\qquad \times \left(p^D(t) - \frac{1}{2}c_i\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) - a_i\right) \\
&\qquad + \beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right] \\
&\quad \times \left(\frac{\sigma^2}{\sum_{j=1}^n \beta_j^R} + \left(\sum_{j\neq i} \tau_j S_i^D(p^D(t)) - (c_i - \tau_i)\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) + \mu - a_i\right)^2\right).
\end{aligned}$$

We can now proceed with an analysis that is very similar to the basic SFE analysis, except that the profit is changed by $\Delta\pi_i(q_i(t), S_j(p^D(t)), j \neq i)$ and obtain a set of simultaneous optimality conditions that characterize the equilibrium and which can be (in principle) transformed into a vector differential equation.

In particular, differentiating the expression for profit with respect to price p_t^D at time t and setting it equal to zero, and writing $S_j^{D\prime}$ for the derivative of S_j^D, we obtain:

$$\begin{aligned}
0 &= \left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) \\
&\quad + \left[c_i\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) + a_i - p^D(t)\right]\left(\gamma^D + \sum_{j\neq i} S_j^{D\prime}(p^D(t))\right) \\
&\quad +2\beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right] \\
&\quad \times\left(\sum_{j\neq i}\tau_j S_j^D(p^D(t)) - (c_i - \tau_i)\left(N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))\right) + \mu - a_i\right) \\
&\quad \times\left(\sum_{j\neq i}\tau_j S_j^{D\prime}(p^D(t)) + (c_i - \tau_i)\left(\gamma^D + \sum_{j\neq i} S_j^{D\prime}(p^D(t))\right)\right).
\end{aligned}$$

Now we observe that $S_i^D(p^D(t)) = N(t) - \gamma^D p^D(t) - \sum_{j\neq i} S_j^D(p^D(t))$ and obtain:

$$\begin{aligned}
0 &= S_i^D(p^D(t)) + [c_i S_i^D(p^D(t)) + a_i - p^D(t)]\left(\gamma^D + \sum_{j\neq i} S_j^{D\prime}(p^D(t))\right) \\
&\quad +2\beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right]\left(\sum_{j\neq i}\tau_j S_j^D(p^D(t)) - (c_i - \tau_i) S_i^D(p^D(t)) + \mu - a_i\right) \\
&\quad \times\left(\sum_{j\neq i}\tau_j S_j^{D\prime}(p^D(t)) + (c_i - \tau_i)(\gamma^D + \sum_{j\neq i} S_j^{D\prime}(p^D(t)))\right).
\end{aligned}$$

This form has extra terms compared to the conditions for the day-ahead equilibrium omitting the real-time market. The extra terms pose some difficulties for transforming the conditions into the form of a differential equation.

To find at least one solution, we will postulate an affine solution of the form:

$$\forall p^D, S_j^D(p^D) = \beta_j^D(p^D - a_i).$$

Unfortunately, this affine solution will not exist unless we additionally assume that the intercepts a_i are the same for each firm. Although this assumption is not realistic, it will at least enable us to find an affine solution. If the a_i are the same for each firm then $\mu = a_i$. Substituting for the assumed affine functional form into the above conditions and dividing by $(p^D(t) - a_i)$ yields:

$$\begin{aligned}
0 &= \beta_i^D + [c_i\beta_i^D - 1]\left(\gamma^D + \sum_{j\neq i}\beta_j^D\right) \\
&\quad +2\beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right] \\
&\quad \times\left(\sum_{j\neq i}\tau_j\beta_j^D - (c_i - \tau_i)\beta_i^D\right)\left(\sum_{j\neq i}\tau_j\beta_j^D + (c_i - \tau_i)(\gamma^D + \sum_{j\neq i}\beta_j^D)\right). \qquad (1)
\end{aligned}$$

Collecting this condition together for all firms yields a set of non-linear simultaneous equations. I have not checked that a solution exists, but since it is a perturbation of the conditions for the standard affine SFE, it seems that solutions should exist at least when the effect of the real-time market on the day-ahead market is relatively small, such as, for example, when the firms are nearly symmetric.

6 Sketch of comparative statics analysis

A key question is how the existence of a real-time market might affect the day-ahead market. This is analogous to the analysis of Allez and Vila. We now sketch how to compare the slopes of the affine solution described in the previous section to those of the standard affine SFE.

We first note that if there was no day-ahead market, the conditions for the day-ahead market would be the same as if we set the real-time supply function slopes $\beta_i^R = 0, \forall i$ in the conditions above. If we assume that the the real-time market has a relatively small effect on the day-ahead market then we can analyze this effect by considering the sensitivity of the day-ahead supply function slopes to the real-time slopes.

Collecting the conditions (1) on the β_i^D and β_i^R together into a vector of non-linear simultaneous equations, we have that:

$$g(\beta^D, \beta^R) = \mathbf{0},$$

where:

- $\beta^D = \begin{bmatrix} \beta_1^D \\ \vdots \\ \beta_n^D \end{bmatrix}$,
- $\beta^R = \begin{bmatrix} \beta_1^D \\ \vdots \\ \beta_n^R \end{bmatrix}$, and
- the i-th entry of $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with g_i given by the right hand side of (1).

Applying the implicit function theorem, and interpreting β^D to be an implicit function of β^R, we have that:

$$\frac{\partial \beta^D}{\partial \beta^R} = -\left[\frac{\partial g}{\partial \beta^D}\right]^{-1} \frac{\partial g}{\partial \beta^R}.$$

There are many terms in this expression and we will only consider it in detail for the condition $\beta^R = \mathbf{0}$.

For $j \neq i$,

$$\begin{aligned}
\frac{\partial g_i}{\partial \beta_j^D} &= c_i\beta_i^D - 1 + 2\beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right] \\
&\quad \left(\tau_j\left(\sum_{j\neq i}\tau_j\beta_j^D + (c_i-\tau_i)(\gamma^D + \sum_{j\neq i}\beta_j^D)\right) + \left(\sum_{j\neq i}\tau_j\beta_j^D - (c_i-\tau_i)\beta_i^D\right)(\tau_j + c_i - \tau_i)\right), \\
&= c_i\beta_i^D - 1, \text{ for } \beta^R = \mathbf{0}, \\
&< 0, \\
\frac{\partial g_i}{\partial \beta_j^R} &= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial g_i}{\partial \beta_i^D} &= 1 + c_i\left(\gamma^D + \sum_{j\neq i}\beta_j^D\right) \\
&\quad -2\beta_i^R\left[1 - \frac{\beta_i^R c_i}{2}\right](c_i - \tau_i)\left(\sum_{j\neq i}\tau_j\beta_j^D + (c_i-\tau_i)(\gamma^D + \sum_{j\neq i}\beta_j^D)\right), \\
&= 1 + c_i\left(\gamma^D + \sum_{j\neq i}\beta_j^D\right), \text{ for } \beta^R = \mathbf{0}, \\
&> 0, \\
\frac{\partial g_i}{\partial \beta_i^R} &= 2\left[1 - \beta_i^R c_i\right]\left(\sum_{j\neq i}\tau_j\beta_j^D - (c_i-\tau_i)\beta_i^D\right)\left(\sum_{j\neq i}\tau_j\beta_j^D + (c_i-\tau_i)(\gamma^D + \sum_{j\neq i}\beta_j^D)\right), \\
&= 2\left(\sum_{j\neq i}\tau_j\beta_j^D - (c_i-\tau_i)\beta_i^D\right)\left(\sum_{j\neq i}\tau_j\beta_j^D + (c_i-\tau_i)(\gamma^D + \sum_{j\neq i}\beta_j^D)\right), \text{ for } \beta^R = \mathbf{0}.
\end{aligned}$$

I have not checked out the signs of the sensitivities of the β^D with respect to the β^R, but it seems plausible that they could be either positive or negative.